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The matrix factorisations of the D-model

Ilka Brunner* and Matthias R. Gaberdiel†

Institut für Theoretische Physik, ETH-Hönggerberg
8093 Zürich, Switzerland

Abstract

The fundamental matrix factorisations of the D-model superpotential are found and identified with the boundary states of the corresponding conformal field theory. The analysis is performed for both GSO-projections. We also comment on the relation of this analysis to the theory of surface singularities and their orbifold description.

*E-mail: brunner@itp.phys.ethz.ch
†E-mail: gaberdiel@itp.phys.ethz.ch

1 Introduction

Models with $N = (2,2)$ worldsheet supersymmetry play a major role in various applications of string theory, most notably in the context of Calabi-Yau compactifications, mirror symmetry and the construction of four-dimensional string vacua. The simplest non-trivial theories with superconformal symmetry are the $N = 2$ minimal models which have an ADE classification. These models play a central role in the construction of Gepner models that describe certain Calabi-Yau compactifications at specific points in their moduli space. In addition to their realisation as abstract conformal field theories, the $N = 2$ minimal models also have a description as Landau Ginzburg theories. For the case of the D-model that shall mainly concern us in this paper, the relevant superpotentials are

$$W_D = x^{\frac{n+1}{2}} - xy^2\,, \qquad W_{D'} = x^{\frac{n+1}{2}} - xy^2 - z^2\,. \tag{1.1}$$

The two different choices correspond to the two possible GSO-projections that can be imposed in conformal field theory.

In this paper, we perform a complete analysis of the B-type branes in both of these theories. These can be studied from two points of view: on the one hand, they have a description in terms of boundary states in rational conformal field theory. On the other hand, according to a proposal in unpublished work of Kontsevich, B-type branes in Landau Ginzburg models can be characterised in terms of matrix factorisations of the superpotential. This proposal was discussed in the physics literature in [1, 2, 3, 4, 5, 6, 7, 8]. One would thus expect that the boundary state analysis in conformal field theory should agree with that of the D-branes in the Landau Ginzburg theory. This was for example seen to be the case for the A-type models in [2, 4].

In the following we shall compare in detail the boundary states of the D-type minimal models with the matrix factorisations of the corresponding Landau Ginzburg models. Furthermore, we shall relate the matrix factorisations of the Landau Ginzburg superpotential W to matrix factorisations that appear in the study of singularity theory. Surface singularities can formally be related to Landau Ginzburg theories by studying the locus $W = 0$ as a hypersurface in $\mathbb{C}^3$. The matrix factorisations encode then the geometrical details of the blow-ups necessary to resolve the surface singularity. Interpreted on the singular variety, they also correspond to the elements of Orlov's category D_{Sg} [9]. Our analysis suggests that the Landau-Ginzburg theory, despite having smaller central charge, captures some of the physical properties of the D-branes on these surface singularities that become massless at the singular point.

The paper is organised as follows. In section 2 we review the relation between matrix factorisations and boundary states for the A-type minimal models. As explained in [4], adding a square to the Landau Ginzburg superpotential corresponds to a change in the GSO projection in conformal field theory. In section 3 we construct all B-type boundary states in the D-model for both possible GSO projections. Since the D-model is a rational conformal field theory (with respect to the $N = 2$ superconformal algebra), the boundary states we construct generate *all* superconformal boundary states. In section 4 we discuss the matrix factorisations of the corresponding Landau Ginzburg models, completing the list of factorisations provided in [4]. Once these additional factorisations are considered, we obtain complete agreement with the conformal field theory description. On the other hand, since the latter description is complete (in the above sense), we can conclude that

the matrix factorisations we have found are all the fundamental matrix factorisations for these superpotentials. This is to say, any matrix factorisation must be the direct sum of these fundamental factorisations. Our analysis also confirms the conjecture of [4] regarding the relation between the different GSO-projections in conformal field theory and the addition of a square to the Landau Ginzburg model. Finally, in section 5, we investigate the relation between D-branes on singular surfaces, the corresponding Landau Ginzburg theories and conformal field theory. We also comment on the description of the surface singularity in terms of the singular quotient $\mathbb{C}^2/\Gamma$, where Γ is a subgroup of $SU(2)$. This allows us to use the methods of [10, 11] to describe D-branes on those orbifolds in terms of representations of Γ. We find complete agreement with the results from the matrix factorisation point of view, confirming yet again that we have found all D-branes.

2 Matrix factorisations

Let us briefly review the description of D-branes in terms of matrix factorisations. This approach was proposed in unpublished form by Kontsevich, and the physical interpretation of it was given in [1, 2, 3, 4, 5, 6]; for a good review of this material see for example [8].

According to Kontsevich's proposal, D-branes in Landau Ginzburg models correspond to matrix factorisations of the superpotential $W(x_i)$,

$$d_0\, d_1 = d_1\, d_0 = W\, \mathbf{1}\,, \tag{2.1}$$

where d_0 and d_1 are $r \times r$ matrices with polynomial entries in the x_i. To a matrix factorisation of this form, one can then associate the boundary BRST operator Q of the form

$$Q = \begin{pmatrix} 0 & d_1 \\ d_0 & 0 \end{pmatrix} . \tag{2.2}$$

The spectrum of open string operators consists of polynomial matrices of the same size. It is naturally $\mathbb{Z}_2$ graded, where the bosons ϕ correspond to block-diagonal matrices, while the fermions t are off-diagonal:

$$\phi = \begin{pmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{pmatrix} , \qquad\qquad t = \begin{pmatrix} 0 & t_1 \\ t_0 & 0 \end{pmatrix} . \tag{2.3}$$

To obtain the physical spectrum between two such branes, one restricts to the degree 0 operators satisfying $D\phi = [Q, \phi] = 0$, and identifies operators that differ only by BRST-exact terms. To calculate this cohomology, we can follow the strategy of [4]. A BRST invariant boson has to satisfy

$$d_1\, \phi_1 = \phi_0\, d_1\,, \qquad \text{and} \qquad d_0\, \phi_0 = \phi_1\, d_0\,, \tag{2.4}$$

where ϕ_0 and ϕ_1 are $r \times r$ matrices. This equation can be solved for ϕ_0 as

$$\phi_0 = \frac{1}{W}\, d_1\, \phi_1\, d_0\,, \tag{2.5}$$

provided that the matrix on the right hand side is divisible by W. After moding out Q-exact terms, the bosonic cohomology can therefore be described using ϕ_1 only. The BRST trivial $\tilde{\phi}_1$ are derivatives of fermions, and hence take the form

$$\tilde{\phi}_1 = (Dt)_1 = t_0 d_1 + d_0 t_1 \,. \tag{2.6}$$

For the fermions, the condition for BRST invariance is

$$t_0\, d_1 + d_0\, t_1 = 0 = d_1\, t_0 + t_1\, d_0 \,, \tag{2.7}$$

which can be solved for t_1

$$t_1 = -\frac{1}{W}\, d_1\, t_0\, d_1 \,, \tag{2.8}$$

resulting again in a divisibility condition. An invariant fermion $(\tilde{t}_0, \tilde{t}_1)$ is BRST trivial if $\tilde{t}_0$ can be written as

$$\tilde{t}_0 = -\phi_1\, d_0 + d_0\, \phi_0 \,, \tag{2.9}$$

for a boson (ϕ_0, ϕ_1).

It is easy to see that the spectrum of two factorisations Q and $\hat{Q}$ is the same, if

$$Q = U\, \hat{Q}\, U^{-1} \,, \qquad U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \,, \tag{2.10}$$

provided that U and its inverse, U^{-1}, are polynomial matrices. We therefore identify two such factorisations [12].

For a given factorisation Q, we call the factorisation where the roles of d_0 and d_1 are reversed the *reverse factorisation* Q^r. It is clear from the above discussion that the bosonic spectrum between two factorisations Q_1 and Q_2 and the fermionic spectrum between Q_1 and Q_2^r coincide.

To make contact with conformal field theory, one has to make sure that the $U(1)$ R-symmetry that becomes the $U(1)$ current symmetry of the $N = 2$ superconformal algebra in the IR is preserved [7]. This means that one has to restrict to homogeneous superpotentials and specify a consistent assignment of R-charge in the boundary theory. By (2.1) and (2.2) this implies that Q should have charge one

$$e^{i\lambda R}\; Q(e^{i\lambda q_i} x_i)\; e^{-i\lambda R} = e^{i\lambda}\; Q \,. \tag{2.11}$$

Then one can assign R-charge to the boundary operators by [7]

$$e^{i\lambda R}\; \phi(e^{i\lambda q_i} x_i)\; e^{-i\lambda R} = e^{i\lambda q}\; \phi \,. \tag{2.12}$$

This description of D-branes should then have a direct correspondence in conformal field theory. In particular, the different factorisations (up to the aforementioned equivalence) should correspond to the different B-type boundary states in conformal field theory. The physical boson spectrum described above corresponds then to the topological open string spectrum from the point of view of conformal field theory. The reverse factorisation of a given factorisation corresponds to the anti-brane of the given brane; thus the physical fermion spectrum in the above description corresponds to the topological open string spectrum between a brane and an anti-brane.

In order to illustrate these concepts and set up notation for the rest of the paper, we briefly review the case of the A-type minimal models.

2.1 A-type minimal models

In [2, 4] A-type minimal models were studied from the matrix point of view. For the potential

$$W_A = x^n \tag{2.13}$$

it can be shown that a class of inequivalent matrix factorisations that generate all factorisations are described by the simple rank 1 factorisations $W = x^l x^{n-l}$, where $l = 1, \ldots, n-1$. The corresponding BRST operator is

$$Q_l = \begin{pmatrix} 0 & x^l \\ x^{n-l} & 0 \end{pmatrix} . \tag{2.14}$$

The factorisations Q_l and Q_{n-l} are related to one another by the exchange of d_0 and d_1; they are therefore reverse factorisations of one another.

The corresponding conformal field theory is described by a single $N = 2$ minimal model with $n = k + 2$. (Our conventions are chosen as in [13].) The spectrum of this theory is (after GSO-projection)

$$\mathcal{H}_A = \bigoplus_{[l,m,s]} \left(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[l,m,-s]}\right) . \tag{2.15}$$

The GSO projection chosen here is the analogue of the Type 0A projection. B-type branes are characterised by the gluing conditions

$$\begin{aligned} \left(L_n - \bar{L}_{-n}\right) \|B\rangle\rangle &= 0 \\ \left(J_n + \bar{J}_{-n}\right) \|B\rangle\rangle &= 0 \\ \left(G^\pm_r + i\,\eta\,\bar{G}^\pm_{-r}\right) \|B\rangle\rangle &= 0\,. \end{aligned} \tag{2.16}$$

The corresponding B-type boundary states were constructed some time ago (see for example [14]), and are explicitly given as

$$\|L, S\rangle\rangle = \sqrt{k+2} \sum_{l+s\in 2\mathbb{Z}} \frac{S_{L0S,l0s}}{\sqrt{S_{l0s,000}}}\, |[l, 0, s]\rangle\rangle\,. \tag{2.17}$$

Here $L = 0, 1, \ldots, k$ and $S = 0, 1, 2, 3$. The boundary states with S even (odd) satisfy the gluing conditions with $\eta = +1$ ($\eta = -1$); in the following we shall restrict ourselves to the case $\eta = +1$, and thus to even S.‡ We also note that

$$\|L, S\rangle\rangle = \|k - L, S + 2\rangle\rangle \tag{2.18}$$

and thus there are only $k + 1$ different boundary states with $\eta = +1$ (and $k + 1$ different boundary states with $\eta = -1$). These boundary states therefore account for all the $N = 2$ Ishibashi states. Finally we note that $\|L, S\rangle\rangle$ and $\|L, S+2\rangle\rangle$ are anti-branes of one another (since they differ by a sign in the coupling to the RR sector states).

‡The D-branes corresponding to $\eta = -1$ or S odd preserve a different supercharge at the boundary. The branes that are described by the different matrix factorisations however always preserve the *same* supercharge at the boundary.

It is then suggestive to identify the matrix factorisation corresponding to Q_l (2.14) with the boundary state with label $L = l - 1$ and $S = 0$. The equivalence (2.18) mirrors the factorisation reversal map $l \mapsto n - l$ on the matrix side. Restricting without loss of generality to the range $l \leq n/2$ one can then easily check, both in conformal field theory and the matrix language, that there are l topological states in the open string spectrum on each of these branes (they correspond to the 'bosons' from the matrix point of view), and l topological states in the open string spectrum between brane and anti-brane (the 'fermions' in the matrix description).

From the conformal field theory perspective it is immediately clear that there is another closely related theory, where one uses a type 0B like GSO projection instead of the GSO projection discussed above. The relevant Hilbert space is

$$\mathcal{H}_{A'} = \bigoplus_{[l,m,s]} \left(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[l,m,s]}\right) . \tag{2.19}$$

It was shown in [4] that from the matrix point of view this theory is described by the superpotential

$$W_{A'} = x^n - y^2 . \tag{2.20}$$

Since the D-models that we shall discuss in this paper are closely related to this theory, let us briefly summarise the results of [4] on factorisations and their relation to boundary states. The superpotential $W_{A'}$ has two classes of factorisations: the first class is given by

$$d_0 = \begin{pmatrix} x^l & y \\ -y & -x^{n-l} \end{pmatrix} , \qquad d_1 = \begin{pmatrix} x^{n-l} & y \\ -y & -x^l \end{pmatrix} . \tag{2.21}$$

One can easily see that the factorisation corresponding to l and $n - l$ are equivalent in the sense of (2.10); in the following we shall therefore always take $l \leq n/2$. Since these are reverse factorisations of one another, the corresponding branes have to be their own anti-branes.

These factorisations are just a special class of the tensor product factorisations discussed in [15]. In fact, they consist of one factor coming from the potential x^n, and another one from the 'trivial' factor y^2 whose physical significance has been discussed in [1]. Accordingly, the open string spectrum inherits the tensor product structure: there are $2l$ bosons of the type

$$a_i = \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix} , \qquad a_{l+i} = \begin{pmatrix} 0 & x^i \\ x^{n-2l+i} & 0 \end{pmatrix} , \quad i = 0, \ldots, l-1 , \tag{2.22}$$

where the 2×2 matrices a_j describe ϕ_1, and ϕ_0 is then determined by (2.5). Likewise, there are $2l$ fermions

$$\eta_i = \begin{pmatrix} 0 & x^i \\ x^i & 0 \end{pmatrix} , \qquad \eta_{l+i} = \begin{pmatrix} x^i & 0 \\ 0 & x^{n-2l+i} \end{pmatrix} , \quad i = 0, \ldots, l-1 , \tag{2.23}$$

where the 2×2 matrices η_j describe t_0, and t_1 is then determined by (2.8). In the case that n is even, the factorisation $l = n/2$ is not irreducible. In fact, it is equivalent (in the sense of (2.10)) to the direct sum of the rank 1 factorisation

$$d_0 = (x^{\frac{n}{2}} + y) , \qquad d_1 = (x^{\frac{n}{2}} - y) , \tag{2.24}$$

and its reverse. The physical open string spectrum of these rank 1 factorisations consists of $n/2$ bosons (but no fermions); correspondingly there are $n/2$ fermions (but no bosons) between the factorisation (2.24) and its reverse.

In conformal field theory, the rank 2 factorisations reviewed above correspond to the boundary states [4]

$$\|L,S\rangle\rangle = (2k+4)^{1/4} \sum_{l \text{ even}} \sum_{\nu=0,1} \frac{S_{Ll}}{\sqrt{S_{0l}}}\, e^{-i\pi\nu S}\, |[l,0,2\nu]\rangle\rangle\,. \tag{2.25}$$

Here S is only defined modulo 2, and $\|L,S\rangle\rangle = \|k-L,S\rangle\rangle$. As before, the identification relates the factorisation (2.21) to the boundary state with $L=l-1$ and $S=0$. (Recall that $n=k+2$.) These boundary states do not couple to RR states and are therefore equivalent to their own anti-branes. (This is simply the statement that S is only defined modulo 2.)

If k is even, then the $L=k/2$ boundary state contains two vacua in its open string spectrum and thus can be further resolved. The explicit form of the two resolved boundary states is [14]

$$\|k/2,S\rangle\rangle_{\rm res} = \frac{1}{2}\left(\|k/2,S\rangle\rangle + \sqrt{k+2}\sum_{s=\pm 1} e^{-\frac{\pi i S s}{2}}\, |[k/2,(k+2)/2,s]\rangle\rangle\right)\,, \tag{2.26}$$

where S is now defined mod 4, and $\|k/2,S\rangle\rangle_{\rm res}$ and $\|k/2,S+2\rangle\rangle_{\rm res}$ are anti-branes of one another. The two branes with $S=0$ and $S=2$ then correspond to the matrix factorisation (2.24) and its reverse, respectively.

We shall now construct the boundary states of the various GSO-projections of the D-model; in section 4 we shall discuss the corresponding matrix factorisations and identify the two descriptions.

3 Boundary states for the D-model

The D-model is only defined if the level k is even. As before for the case of the A-type minimal model there are two possible GSO-projections one can consider. We begin by discussing the theory that is analogous to type 0B. Its spectrum depends on whether $k/2$ is even or odd, *i.e.* on whether k is divisible by 4 or not. In the former case, the spectrum is

$$\mathcal{H} = \bigoplus_{[l,m,s],\ l \text{ even}} \Big(\big(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[l,m,s]}\big) \oplus \big(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[k-l,m,s]}\big)\Big)\,. \tag{3.1}$$

On the other hand, if $k/2$ is odd the spectrum is

$$\mathcal{H} = \bigoplus_{[l,m,s],\ l \text{ even}} \Big(\big(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[l,m,s]}\big)\Big) \oplus \bigoplus_{[l,m,s],\ l \text{ odd}} \Big(\big(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[k-l,m,s]}\big)\Big)\,. \tag{3.2}$$

We are interested in the B-type boundary states of this theory. In our conventions, the B-type Ishibashi states are characterised by the gluing conditions (2.16). The structure of these Ishibashi states (and thus the corresponding boundary states) depends on the two cases above, and we shall therefore discuss them in turn.

3.1 The case $k/2$ odd

For $k/2$ odd, the theory (3.2) has the Ishibashi states

$$|[l,0,s]\rangle\rangle \in \mathcal{H}_{[l,0,s]} \otimes \bar{\mathcal{H}}_{[l,0,s]}\,, \tag{3.3}$$

where l and s are both even, and the Ishibashi states

$$|[l,(k+2)/2,s]\rangle\rangle \in \mathcal{H}_{[l,\frac{k+2}{2},s]} \otimes \bar{\mathcal{H}}_{[k-l,\frac{k+2}{2},s]} \tag{3.4}$$

where l and s are both odd. In total there are therefore $2(k+1)$ Ishibashi states of the coset theory; they give rise to $(k+1)$ Ishibashi states of the $N=2$ theory with one spin structure $\eta=+1$, and $(k+1)$ Ishibashi states of the $N=2$ algebra with the other $\eta=-1$. The corresponding boundary states are described by

$$\begin{aligned} \|L,M,S\rangle\rangle &= \sqrt{\frac{2k+4}{2}}\left(\sum_{l,s \text{ even}} \frac{S_{LMS,l0s}}{\sqrt{S_{000,l0s}}}\,|[l,0,s]\rangle\rangle \right. \\ &\qquad \left. + \sum_{l,s \text{ odd}} \frac{S_{LMS,l\frac{k+2}{2}s}}{\sqrt{S_{000,l\frac{k+2}{2}s}}}\,|[l,(k+2)/2,s]\rangle\rangle\right). \end{aligned} \tag{3.5}$$

Here $L+M+S$ is even, and we have the identifications

$$\begin{aligned} \|L,M,S\rangle\rangle &= \|k-L,M+k+2,S+2\rangle\rangle = \|k-L,M,S+2\rangle\rangle \\ &= \|L,M+4,S\rangle\rangle = \|L,M+2,S+2\rangle\rangle\,. \end{aligned} \tag{3.6}$$

For fixed spin structure $\eta=+1$ (corresponding to $S=0,2$), we have for every value of L only two possible values of M, namely $M=0,2$ or $M=1,3$. Furthermore, because of the last identification, of the four choices ($S=0,2,\ M=0,2$) or ($S=0,2,\ M=1,3$) only two describe different boundary states. For $L\neq k/2$, the pair of boundary states with L and $k-L$ account therefore each for two different boundary states giving in total k different boundary states. For $L=k/2$ on the other hand, it is easy to see that the second sum in (3.5) vanishes, and thus $\|k/2,+1,0\rangle\rangle = \|k/2,-1,0\rangle\rangle$, thus giving only one additional boundary state. In total the above construction therefore gives $k+1$ different boundary states of fixed spin structure, and thus accounts for all the Ishibashi states.

These boundary states satisfy the Cardy condition since their overlap equals

$$\begin{aligned} \langle\langle L_1,M_1,S_1\|q^{L_0+\bar{L}_0-\frac{c}{12}}\|L_2,M_2,S_2\rangle\rangle &= \sum_{[l,m,s]} \chi_{(l,m,s)}(\tilde{q})\,\delta^{(2)}(S_2+s-S_1)\times \\ \times\Big(N_{L_2\ l}{}^{L_1}\,\delta^{(4)}&(m-s+M_2-M_1-S_2+S_1) \\ &+N_{L_2\ k-l}{}^{L_1}\,\delta^{(4)}(m-s+M_2-M_1-S_2+S_1+2)\Big)\,. \end{aligned} \tag{3.7}$$

Here $N_{L\ l}{}^{\hat{L}}$ denotes the level k fusion rules of $su(2)$, and $\chi_{(l,m,s)}$ is the character of the coset representation. We note that the right hand side is invariant under the field identification, $(l,m,s)\mapsto(k-l,m+k+2,s+2)$, as it must be.

3.1.1 Topological spectrum

It is now straightforward to read off from this formula the topological spectrum between two such branes. If we consider only D-branes of a fixed spin structure, we may set, without loss of generality, $S=0$. Then we can restrict the open string sum without loss of generality to the states with $s=0$; then the topological states arise for $l=m$.
For example, for the case $L=L_1=L_2$ (and $S_1=S_2=0$), the number of topological states T is

$$M_1 = M_2 \quad \text{(bosons)}: \qquad |T| = \begin{cases} L+2 & \text{if } L \text{ is even} \\ L+1 & \text{if } L \text{ is odd,} \end{cases} \tag{3.8}$$

and

$$M_1 = M_2+2 \quad \text{(fermions)}: \qquad |T| = \begin{cases} L & \text{if } L \text{ is even} \\ L+1 & \text{if } L \text{ is odd.} \end{cases} \tag{3.9}$$

Note that the boundary state corresponding to $L=k/2$ (which is an odd number) has the same number of topological fermions and bosons in its open string spectrum.

3.2 The case $k/2$ even

For $k/2$ even, the theory (3.1) has the $(k+2)$ Ishibashi states

$$|[l,0,s]\rangle\!\rangle \in \mathcal{H}_{[l,0,s]} \otimes \bar{\mathcal{H}}_{[l,0,s]}\,, \tag{3.10}$$

where l and s are both even, and the $(k+2)$ Ishibashi states

$$|[l,(k+2)/2,s]\rangle\!\rangle \in \mathcal{H}_{[l,\frac{k+2}{2},s]} \otimes \bar{\mathcal{H}}_{[k-l,\frac{k+2}{2},s]}\,, \tag{3.11}$$

where l is even and s is odd. In addition, there are the two Ishibashi states from the first sum in (3.1)

$$|[k/2,(k+2)/2,s\rangle\!\rangle \in \mathcal{H}_{[k/2,\frac{k+2}{2},s]} \otimes \bar{\mathcal{H}}_{[k/2,\frac{k+2}{2},s]}\,, \qquad s=\pm 1 \tag{3.12}$$

and the two Ishibashi states from the second sum in (3.1)

$$|[k/2,0,s\rangle\!\rangle \in \mathcal{H}_{[k/2,0,s]} \otimes \bar{\mathcal{H}}_{[k-k/2,0,s]}\,, \qquad s=0,2\,. \tag{3.13}$$

In total there are therefore $2(k+4)$ Ishibashi states of the coset theory; they give rise to $(k+4)$ Ishibashi states of the $N=2$ theory with one spin structure $\eta=+1$, and $(k+4)$ Ishibashi states of the $N=2$ algebra with the other $\eta=-1$.
The corresponding boundary states are described by

$$\begin{aligned} \|L,M,S\rangle\!\rangle \;=\; \sqrt{\frac{2k+4}{2}} \sum_{l \text{ even}} \Bigg(& \sum_{s \text{ even}} \frac{S_{LMS,l0s}}{\sqrt{S_{000,l0s}}} \, |[l,0,s]\rangle\!\rangle \\ & + \sum_{s \text{ odd}} \frac{S_{LMS,l\frac{k+2}{2}s}}{\sqrt{S_{000,l\frac{k+2}{2}s}}} \, |[l,(k+2)/2,s]\rangle\!\rangle \Bigg)\,. \end{aligned} \tag{3.14}$$

Here $L+M+S$ is even, and $L \neq k/2$. We have the identifications

$$\begin{aligned} \|L,M,S\rangle\!\rangle &= \|k-L,M+k+2,S+2\rangle\!\rangle = \|k-L,M,S\rangle\!\rangle \\ &= \|L,M+4,S\rangle\!\rangle = \|L,M+2,S+2\rangle\!\rangle\,. \end{aligned} \tag{3.15}$$

For fixed spin structure $\eta = +1$ (corresponding to $S = 0,2$), we have for every value of L only two possible values of M, namely $M = 0,2$ or $M = 1,3$. Furthermore, because of the last identification, of the four choices ($S = 0,2,\ M = 0,2$) or ($S = 0,2,\ M = 1,3$) only two describe different boundary states. For $L \neq k/2$, the pair of boundary states with L and $k-L$ account therefore each for two different boundary states giving in total k different boundary states. The remaining four boundary states correspond to the 'resolved' boundary states for $L = k/2$, and will be described shortly.

These boundary states satisfy the Cardy condition since their overlap equals

$$\begin{aligned} \langle\!\langle L_1,M_1,S_1\| q^{L_0+\bar{L}_0-\frac{c}{12}}\, \|L_2,M_2,S_2\rangle\!\rangle = \sum_{[l,m,s]} \chi_{(l,m,s)}(\tilde{q})\, \delta^{(2)}(S_2+s-S_1) \times \\ \times\Big(N_{L_2\ l}{}^{L_1} + N_{L_2\ k-l}{}^{L_1}\Big)\, \delta^{(4)}(m-s+M_2-M_1-S_2+S_1)\,. \end{aligned} \tag{3.16}$$

We observe again that the right hand side is invariant under the field identification, $(l,m,s) \mapsto (k-l, m+k+2, s+2)$, as it must be.

It is easy to see that the topological open string spectrum of these branes is described by the same formulae as (3.8) and (3.9) above.

3.2.1 The resolved branes

The remaining boundary states are of the form

$$\begin{aligned} \|k/2,M,S,\pm\rangle\!\rangle &= \frac{1}{2}\, \|k/2,M,S\rangle\!\rangle_{(3.14)} \\ &\quad \pm\frac{\sqrt{2k+4}}{4}\Big(\sum_{s\ \mathrm{odd}} e^{i\pi M/2}\, e^{-i\pi s S/2}\, |[k/2,(k+2)/2,s]\rangle\!\rangle \\ &\qquad + \sum_{s\ \mathrm{even}} e^{-i\pi s S/2}\, |[k/2,0,s]\rangle\!\rangle\Big)\,. \end{aligned} \tag{3.17}$$

Here $M+S$ is even, and we have the same identifications as in the second line of (3.15). Thus these boundary states account for four more boundary states (of a fixed spin structure).

One easily checks that the overlaps of these boundary states with the previously constructed boundary states is simply

$$\begin{aligned} \langle\!\langle L_1,M_1,S_1\| q^{L_0+\bar{L}_0-\frac{c}{12}}\, \|k/2,M_2,S_2,\pm\rangle\!\rangle = \sum_{[l,m,s]} \chi_{(l,m,s)}(\tilde{q})\, \delta^{(2)}(S_2+s-S_1) \times \\ \times N_{k/2\ l}{}^{L_1}\, \delta^{(4)}(m-s+M_2-M_1-S_2+S_1)\,. \end{aligned} \tag{3.18}$$

On the other hand, the overlaps involving two resolved branes is

$$\begin{aligned} &\langle\!\langle k/2,M_1,S_1,\pm\| q^{L_0+\bar{L}_0-\frac{c}{12}}\, \|k/2,M_2,S_2,\pm\rangle\!\rangle \\ &= \sum_{[l,m,s]\ l=4n} \chi_{(l,m,s)}(\tilde{q})\, \delta^{(2)}(S_2+s-S_1)\, \delta^{(4)}(m-s+M_2-M_1-S_2+S_1)\,, \end{aligned} \tag{3.19}$$

and

$$
\begin{aligned}
&\langle\!\langle k/2, M_1, S_1, \pm \| q^{L_0+\bar{L}_0-\frac{c}{12}} \, \| k/2, M_2, S_2, \mp \rangle\!\rangle && (3.20)\\
&= \sum_{[l,m,s] \;\; l=4n+2} \chi_{(l,m,s)}(\tilde{q}) \, \delta^{(2)}(S_2+s-S_1) \, \delta^{(4)}(m-s+M_2-M_1-S_2+S_1) \, .
\end{aligned}
$$

It is now straightforward to determine the topological open string spectrum of these branes. As before, we may set, without loss of generality (if we restrict ourselves to a specific spin structure) $S_1 = S_2 = 0$. Then the number of topological states on the +brane is

$$
M_1 = M_2 \quad \text{(bosons)}: \qquad |T| = \frac{k}{4} + 1 \, , \tag{3.21}
$$

$$
M_1 = M_2 + 2 \quad \text{(fermions)}: \qquad |T| = 0 \, . \tag{3.22}
$$

Obviously the result for the −brane is identical. On the other hand, the topological spectrum between the +brane and the −brane is

$$
M_1 = M_2 : \qquad |T| = 0 \, , \tag{3.23}
$$

$$
M_1 = M_2 + 2 : \qquad |T| = \frac{k}{4} \, . \tag{3.24}
$$

3.3 The other GSO-projection

As we mentioned before, we can also consider the 0A-like GSO-projection. Then the spectrum of the D-model is for $k/2$ even

$$
\mathcal{H} = \bigoplus_{[l,m,s], \; l \text{ even}} \Big(\big(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[l,m,-s]}\big) \oplus \big(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[k-l,m,-s]}\big) \Big) \, . \tag{3.25}
$$

On the other hand, if $k/2$ is odd, the spectrum is

$$
\mathcal{H} = \bigoplus_{[l,m,s], \; l \text{ even}} \Big(\big(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[l,m,-s]}\big) \Big) \oplus \bigoplus_{[l,m,s], \; l \text{ odd}} \Big(\big(\mathcal{H}_{[l,m,s]} \otimes \bar{\mathcal{H}}_{[k-l,m,-s]}\big) \Big) \, . \tag{3.26}
$$

In both cases we have the $k+2$ Ishibashi states

$$
|[l,0,s]\rangle\!\rangle \in \mathcal{H}_{[l,0,s]} \otimes \bar{\mathcal{H}}_{[l,0,-s]} \, , \tag{3.27}
$$

where l and s are both even, as well as the two Ishibashi states

$$
|[k/2,0,s]\rangle\!\rangle \in \mathcal{H}_{[k/2,0,s]} \otimes \bar{\mathcal{H}}_{[k/2,0,-s]} \, , \qquad k/2 + s = \text{even} \, . \tag{3.28}
$$

We therefore expect to have $k/2+2$ boundary states of each spin structure. Some of these boundary states are given by

$$
\|L, S\rangle\!\rangle = \sqrt{2k+4} \sum_{l,s \,\text{even}} \frac{S_{L0S,l0s}}{\sqrt{S_{000,l0s}}} \, |[l,0,s]\rangle\!\rangle \, , \tag{3.29}
$$

where $L = 0, 1, \ldots, k$ with $L \neq k/2$ and S is defined modulo 4. We observe that

$$\|L, S\rangle\!\rangle = \|k - L, S\rangle\!\rangle = \|L, S + 2\rangle\!\rangle \,. \tag{3.30}$$

For fixed spin structure (say S even) there are thus $k/2$ such boundary states. Their overlap equals

$$\begin{aligned}&\langle\!\langle L_1, S_1 \| q^{L_0 + \bar{L}_0 - \frac{c}{12}} \, \|L_2, S_2\rangle\!\rangle \\ &\qquad = \sum_{[l,m,s]} \chi_{(l,m,s)}(\tilde{q}) \, \delta^{(2)}(S_2 + s - S_1) \Big(N_{L_2 \; l}{}^{L_1} + N_{L_2 \; k-l}{}^{L_1} \Big) \,.\end{aligned}$$

Thus the open string spectrum on the brane $\|L, S\rangle\!\rangle$ has $2(L+1)$ topological states; the same is true for the open string between the $\|L, S\rangle\!\rangle$ brane and its anti-brane.
The remaining two boundary states are then

$$\|k/2, S, \pm\rangle\!\rangle = \frac{1}{2} \|k/2, S\rangle\!\rangle_{(3.29)} \pm \sqrt{\frac{k+2}{4}} \sum_s e^{-i\pi S s/2} \, |[k/2, 0, s]\rangle\!\rangle \,, \tag{3.31}$$

where the sum over s runs over $0, 2$ if $k/2$ is even, and over ± 1 if $k/2$ is odd. For $k/2$ odd, these two branes are anti-branes of one another, *i.e.* $\|k/2, S, +\rangle\!\rangle = \|k/2, S+2, -\rangle\!\rangle$, and their overlaps equal

$$\begin{aligned}&\langle\!\langle k/2, S_1, \pm \| q^{L_0 + \bar{L}_0 - \frac{c}{12}} \, \|k/2, S_2, \pm\rangle\!\rangle \\ &\qquad = \sum_{[l,m,s],\, l \,\text{even}} \chi_{(l,m,s)}(\tilde{q}) \, \delta^{(2)}(S_2 + s - S_1) \, \delta^{(4)}(l + S_2 + s - S_1) \,.\end{aligned}$$

There are then $(k+2)/4$ topological states in the open string spectrum of each of these branes, and $(k+2)/4$ topological states in the open string spectrum between the brane and the anti-brane.

For $k/2$ even, on the other hand, each of the two branes $\|k/2, S, \pm\rangle\!\rangle = \|k/2, S+2, \pm\rangle\!\rangle$ is its own anti-brane. In this case their overlaps equal

$$\begin{aligned}&\langle\!\langle k/2, S_1, \pm \| q^{L_0 + \bar{L}_0 - \frac{c}{12}} \, \|k/2, S_2, \pm\rangle\!\rangle \\ &\qquad = \sum_{[l,m,s]} \chi_{(l,m,s)}(\tilde{q}) \, \delta^{(2)}(S_2 + s - S_1) \, \delta^{(4)}(l) \,,\end{aligned}$$

$$\begin{aligned}&\langle\!\langle k/2, S_1, \pm \| q^{L_0 + \bar{L}_0 - \frac{c}{12}} \, \|k/2, S_2, \mp\rangle\!\rangle \\ &\qquad = \sum_{[l,m,s]} \chi_{(l,m,s)}(\tilde{q}) \, \delta^{(2)}(S_2 + s - S_1) \, \delta^{(4)}(l+2) \,.\end{aligned}$$

In this case there are $k/4 + 1$ topological states in the open string spectrum of either of these two branes, and $k/4$ topological states in the open string between the two different branes.

4 The matrix factorisation description

We now want to describe the matrix factorisations that correspond to the above boundary states. As for the case of the A-type minimal model, the two different GSO-projections correspond to two different superpotentials. We shall now discuss them in turn.

4.1 The first D-model

The theory whose boundary states we discussed in section 3.1 and 3.2 corresponds to the superpotential

$$W_D = x^{n+1} - xy^2 \,, \tag{4.1}$$

where $n = k/2$. Its matrix factorisations have partially been studied in [4], but, as we shall see, the list of factorisations given there is incomplete. If we include another class of factorisations, we obtain perfect agreement with the conformal field theory results described above. Given that these boundary states generate all boundary states of the superconformal field theory, we can turn the argument around and conclude that we have identified all fundamental factorisations for this potential, *i.e.* that any factorisation is equivalent to a direct sum of these fundamental factorisations.

4.1.1 Rank 1 factorisations

We start with a discussion of the rank 1 factorisations, all of which have been discussed in [4]. The most obvious factorisation is

$$\mathcal{R}_0 : \qquad d_0 = x^n - y^2, \quad d_1 = x \,, \tag{4.2}$$

as well as its reverse. The bosonic spectrum of either factorisation consists of two states, whereas the fermionic spectrum is empty. The only boundary states with two bosons and no fermions in their open string spectrum are the $L = 0$ and $L = k$ boundary states of (3.5) and (3.14), respectively. They must therefore correspond to the above factorisation (and its reverse).

Generically this is the only rank 1 factorisation, but if n is even, there are in addition the two resolved rank 1 factorisations, given by

$$\begin{aligned} \mathcal{R}_+ \quad &: \qquad d_0 = (x^{n/2} + y) \,, \qquad d_1 = x(x^{n/2} - y) \,, \\ \mathcal{R}_- \quad &: \qquad d_0 = x(x^{n/2} + y) \,, \qquad d_1 = (x^{n/2} - y) \,, \end{aligned}$$

as well as their reverses. Either of these factorisations has a purely bosonic spectrum consisting of $n/2 + 1$ states, and there are $n/2$ fermionic operators propagating between $\mathcal{R}_+$ and $\mathcal{R}_-$. Given that $n/2 = k/4$, this matches precisely with the spectrum of the resolved branes described in section 3.2.1 that also only exist provided that $k/2$ is even.

4.1.2 Rank 2 factorisations

Next we turn to the rank 2 factorisations. In [4] the factorisations of the form

$$d_0 = \begin{pmatrix} x^l & \alpha \\ -\beta & -x^{n+1-l} \end{pmatrix} , \quad d_1 = \begin{pmatrix} x^{n+1-l} & \alpha \\ -\beta & -x^l \end{pmatrix} , \quad \alpha\beta = xy^2 \tag{4.3}$$

were considered. Exchanging l with $n+1-l$ amounts to exchanging d_0 and d_1, and hence to considering the reverse factorisation. The same holds for the exchange of α and β. The only two inequivalent choices for α, β are therefore $\beta = x$ or $\beta = y$. As was shown in [4]

the choice $\beta = x$ leads to a class of factorisations that is equivalent to the direct sum of the factorisation (4.2) and its reverse. This leaves us with the factorisations

$$\mathcal{S}_l : \qquad d_0 = \begin{pmatrix} x^l & xy \\ -y & -x^{n+1-l} \end{pmatrix} \quad d_1 = \begin{pmatrix} x^{n+1-l} & xy \\ -y & -x^l \end{pmatrix} , \tag{4.4}$$

where $l = 1, \ldots, n$. One easily finds that the factorisation $\mathcal{S}_l$ has $2l$ bosons and $2l$ fermions. Comparing with the conformal field theory analysis, one would then like to identify the factorisation $\mathcal{S}_l$ with the boundary state (3.5) or (3.14) with $L = 2l - 1$. Note that $l \mapsto n + 1 - l$, which maps the factorisation to the reverse factorisation, then corresponds to $L = 2l - 1 \mapsto k - L$, which maps the brane to the anti-brane. [If $k/2$ is even, the anti-brane of $\|L, M, 0\rangle\!\rangle$ is $\|k - L, M, 2\rangle\!\rangle$, while for $k/2$ odd it is $\|k - L, M, 0\rangle\!\rangle$.]

As is clear from the conformal field theory analysis, the above factorisations do not yet account for all branes since there are also the boundary states (3.5) or (3.14) with L even. The corresponding class of factorisations can be directly obtained from the factorisations (2.21) of the A-type minimal model with potential $W_{A'} = x^n - y^2$: since $W_D = x(x^n - y^2) = xW_{A'}$, one obtains a factorisation for the D-model by multiplying the matrix d_1 of any A-model factorisation by x. This leads to the following factorisations

$$\mathcal{T}_l : \qquad d_0^D = \begin{pmatrix} x^l & y \\ -y & -x^{n-l} \end{pmatrix} = d_0^{A'} , \quad d_1^D = \begin{pmatrix} x^{n-l+1} & xy \\ -xy & -x^{l+1} \end{pmatrix} = x d_1^{A'} . \tag{4.5}$$

Here $l = 0, \ldots, n$, and the factorisation $\mathcal{T}_l$ is equivalent to $\mathcal{T}_{n-l}$, but neither is equivalent to $\mathcal{T}_l^r$. We now propose that the factorisation $\mathcal{T}_l$ corresponds to the boundary state (3.5) or (3.14) with $L = 2l$. To confirm this proposal, we have to determine the spectrum of these factorisations.

The calculation of the fermionic part of the spectrum proceeds exactly as in the case of the A-model. For this, consider a fermion described by its two components (t_0, t_1). BRST-invariance means that

$$t_0 \, d_1^D = -d_0^D \, t_1 , \tag{4.6}$$

which can be solved by

$$t_1 = -\frac{1}{W_D} \, d_1^D \, t_0 \, d_1^D . \tag{4.7}$$

This requires, of course, that the entries of the matrix appearing on the right hand side are all divisible by W_D. In terms of the A-model data, this equation can be rewritten as

$$t_1 = -x \, \frac{x}{W_D} \, d_1^{A'} \, t_0 \, d_1^{A'} = -x \, \frac{1}{W_{A'}} \, d_1^{A'} \, t_0 \, d_1^{A'} , \tag{4.8}$$

where W_D/x is the A-model potential $W_{A'}$. Without the additional factor of x on the right hand side, this condition is simply the A-model result, implying a divisibility condition for the entries of the matrix $d_1^{A'} \, t_0 \, d_1^{A'}$. Since x does not divide $W_{A'}$ the divisibility conditions of the A- and D-model are therefore equivalent.
Any solution for t_0 is BRST-trivial if

$$t_0 = -\phi_1 \, d_0^D + d_0 \, \phi_0^D . \tag{4.9}$$

Since $d_0^D = d_0^{A'}$ it is obvious that the fermionic part of the spectrum for the A- and D-model is the same for this class of factorisations. Thus there are $2l$ fermions for the factorisation $\mathcal{T}_l$.

Let us now turn to the bosonic spectrum. The condition for an operator (ϕ_0, ϕ_1) to be BRST invariant is

$$d_1^D \, \phi_1 = \phi_0 \, d_1^D \,, \tag{4.10}$$

which is the same condition as in the A-model. Solving for ϕ_0, one obtains the divisibility condition

$$\phi_0^D = \frac{1}{W_D} \, d_1^D \, \phi_1 \, d_0^D = \frac{1}{W_{A'}} \, d_1^{A'} \, \phi_1 \, d_0^{A'} \,, \tag{4.11}$$

which is explicitly (since $\mathcal{T}_l \equiv \mathcal{T}_{n-l}$ we may assume that $l \leq n/2$)

$$y(\phi_1^{11} - \phi_1^{22}) + x^l \phi_1^{21} - x^{n-l} \phi_1^{12} = 0 \qquad \text{mod } W_{A'} \,, \tag{4.12}$$

where ϕ_1^{ij} denote the matrix elements of the 2×2 matrix ϕ_1. The possible solutions for ϕ_1 are further restricted by the requirement that the operator is BRST non-trivial. BRST trivial operators can be written in the form (see (2.6))

$$\begin{aligned}
\phi_1^{11} &= t_0^{11} \, x^{n-l+1} - t_0^{12} \, xy + x^l \, t_1^{11} + y \, t_1^{21} && (4.13)\\
\phi_1^{12} &= t_0^{11} xy - t_0^{12} \, x^{l+1} + x^l \, t_1^{12} + y \, t_1^{22} && (4.14)\\
\phi_1^{21} &= t_0^{21} x^{n-l+1} - t_0^{22} \, xy - y \, t_1^{11} - x^{n-l} \, t_1^{21} && (4.15)\\
\phi_1^{22} &= t_0^{21} xy - t_0^{22} \, x^{l+1} - y \, t_1^{12} - x^{n-l} \, t_1^{22} \,. && (4.16)
\end{aligned}$$

One can now use the freedom in t_1 to restrict the powers of x appearing in the representatives of the BRST cohomology classes to lie in the range $0, \ldots, l-1$ (for ϕ_1^{11} and ϕ_1^{12})) and $0, \ldots, n-l-1$ (for ϕ_1^{21} and ϕ_1^{22})). On the other hand, this then uses up the freedom described by t_1, and in particular, the y-dependence of the solution cannot be eliminated any longer. Indeed, in addition to the obvious x-dependent solutions (that are as for the A-model, see (2.22))

$$a_i = \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix} , \quad a_{l+i} = \begin{pmatrix} 0 & x^i \\ x^{n-2l+i} & 0 \end{pmatrix} , \quad i = 0, \ldots, l-1 \,, \tag{4.17}$$

there are now the following two non-trivial solutions (for ϕ_1)

$$b_1 = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} , \qquad \text{and} \qquad b_2 = \begin{pmatrix} 0 & y \\ yx^{n-2l} & 0 \end{pmatrix} . \tag{4.18}$$

This means, in particular, that there are two more bosons than fermions for each factorisation $\mathcal{T}_l$. This is then in perfect agreement with the conformal field theory spectra (3.8) and (3.9) for the branes with $L = 2l$.

In summary, we therefore propose the identifications:

Boundary state	Matrix factorisation	
$\|0, M, S\rangle\rangle$	$\mathcal{R}_0 \, , \mathcal{R}_0^r$	
$\|2l-1, M, S\rangle\rangle$	$\mathcal{S}_l \, , \mathcal{S}_l^r$	$l = 1, \ldots, [(n+1)/2]$
$\|2l, M, S\rangle\rangle$	$\mathcal{T}_l \, , \mathcal{T}_l^r$	$l = 0, \ldots, [n/2]$
$\|k/2, M, S, \pm\rangle\rangle$	$\mathcal{R}_\pm \, , \mathcal{R}_\pm^r$	n even

As a consistency check we note that we have now identified the boundary state with $L = 0$ with two factorisations, namely with $\mathcal{R}_0$ and with $\mathcal{T}_0$. Thus we need to have that

these two factorisations are actually equivalent (in the sense of (2.10)), and this is indeed easily checked. Furthermore, for $k/2 = n$ even, the boundary state $\|k/2, M, S\rangle\rangle$ is not fundamental, but can be resolved as explained in section 3.2.1. We have identified these resolved branes with the factorisations $\mathcal{R}_\pm$; thus we expect that for n even the factorisation $\mathcal{T}_{n/2}$ is equivalent to the direct sum of $\mathcal{R}_+$ and $\mathcal{R}_-$, and this is also straightforwardly confirmed.

Another consistency check concerns the various flows of D-brane configurations. By switching on suitable tachyons, one can show that there are flows from the point of view of the matrix factorisation (such flows were first discussed, for the case of the A-model, in [16])

$$\begin{aligned} \mathcal{R}_0 \oplus \mathcal{R}_0^r &\rightarrow \mathcal{S}_1 \\ \mathcal{S}_l \oplus \mathcal{R}_0 &\rightarrow \mathcal{T}_l \\ \mathcal{T}_l \oplus \mathcal{R}_0^r &\rightarrow \mathcal{S}_{l+1} \,. \end{aligned}$$

These flows are easily seen to be compatible with the RR-charges of the corresponding boundary states. They are also in agreement with what one expects based on the analysis of [17]. Finally, we note that the $L = 0$-brane (together with the two resolved branes if n is even) generates all D-brane charges since all D-branes (except for the resolved branes) can be obtained from it by the above flows. The corresponding matrix factorisations of these three D-branes all have rank 1. This is similar to the situation for the tensor product of two A-models where the permutation branes (whose factorisations also have rank 1) generate all the charges [13].

4.2 The other D-model

Finally we consider the superpotential

$$W_{D'} = x^{n+1} - xy^2 - z^2 \tag{4.19}$$

that should correspond, following the logic of [4], to the D-model with the other GSO-projection that was discussed in section 3.3. The easiest class of factorisations of this model are of tensor product type: for any factorisation of W_D one obtains a factorisation of $W_{D'}$, where one factorises z^2 as $z^2 = zz$. Since the spectrum on a brane with superpotential $W = z^2$ consists of the identity $\phi_0 = \phi_1 = 1$ and a fermion $t_0 = -t_1 = 1$, it is very simple to calculate the spectrum of these factorisations. The number of bosons is simply given by the sum of the number of bosons and fermions of the corresponding factorisation of the D-model W_D. The same holds for the number of fermions. In particular, the number of bosons always equals the number of fermions for any factorisation of this type.

We now propose that we can identify the boundary states (3.29) with these tensor product factorisations. As is explained in section 3.3, each of these branes has $2(L+1)$ topological states in its open string spectrum (irrespective of whether L is even or odd), and the same is true for the open string between the brane and its anti-brane. This then matches precisely with the above spectrum of the corresponding tensor product factorisations, given our previous identifications of the boundary states with the matrix factorisations for the W_D model: in particular, the sum of the number of bosons and fermions in (3.8) and (3.9) is precisely equal to $2(L+1)$ for all L.

The analysis works similarly for the two resolved branes $\|k/2, S, \pm\rangle\rangle$ for $k/2$ even. This leaves us with identifying the resolved brane $\|k/2, S, \pm\rangle\rangle$ for $n = k/2$ odd, which does not come from such a tensor factorisation. For n odd there are however additional factorisations of the form

$$d_0 = \begin{pmatrix} x^{\frac{n+1}{2}} - z & \alpha \\ -\beta & -(x^{\frac{n+1}{2}} + z) \end{pmatrix}, \quad d_1 = \begin{pmatrix} x^{\frac{n+1}{2}} + z & \alpha \\ -\beta & -(x^{\frac{n+1}{2}} - z) \end{pmatrix}, \quad \alpha\beta = xy^2 . \tag{4.20}$$

Exchanging α and β maps the factorisation to its reverse, and thus effectively the only inequivalent choices are $\beta = x$ and $\beta = y$. As before for the case of the D-model, one may expect that $\beta = x$ is in some sense trivial, and indeed one can show that this factorisation is equivalent to the tensor product factorisation corresponding to the D-model factorisation $\mathcal{R}_0$,

$$d_0 = \begin{pmatrix} x^n - y^2 & z \\ -z & -x \end{pmatrix}, \quad d_1 = \begin{pmatrix} x & z \\ -z & -(x^n - y^2) \end{pmatrix}. \tag{4.21}$$

On the other hand, the topological spectrum of the factorisation (4.20) with $\beta = y$ has $(n + 1)/2$ bosonic and fermionic states, which thus agrees with the result of section 3.3.

We have therefore also managed to identify the boundary states of this D-model with the factorisations of the superpotential $W_{D'}$. Since we know that the boundary states we have described generate all possible boundary states, the same must be true for the above factorisations.

5 Matrix factorisations and singularity theory

Matrix factorisations are also a well-known tool in the theory of singularities in mathematics. In the case of complex dimension two, the simple singularities are known to have an ADE classification. The relevant singularities are described by hypersurfaces in $\mathbb{C}^3$ characterised by the equations[§]

$$\begin{aligned} A_{n-1}: &\quad W = x^n - yz = 0 \\ D_{n+2}: &\quad W = x^{n+1} - xy^2 - z^2 = 0 \\ E_6: &\quad W = x^4 + y^3 + z^2 = 0 \\ E_7: &\quad W = yx^3 + y^3 + z^2 = 0 \\ E_8: &\quad W = x^5 + y^3 + z^2 = 0 . \end{aligned} \tag{5.1}$$

The resolution of the singularities is obtained by blow-ups, where each node of the associated ADE Dynkin diagram corresponds to an exceptional divisor. For the case of these simple surface singularities, the explicit description of the blow-up can be encoded in a matrix factorisation of the polynomial W defining the hypersurface [18, 19, 20].

D-branes on such singular geometries have recently been discussed in [9]. There it was proposed to define a category D_{Sg} that is supposed to capture the topological sector of B-type branes on singular spaces. On smooth manifolds, any coherent sheaf has a finite resolution of locally free sheaves of finite type. On singular spaces this is no longer the case and Orlov defined D_{Sg} to be the quotient of the bounded derived category of coherent

[§]By a change of variables one can also rewrite the first equation as $W = x^n - y^2 - z^2 = 0$.

sheaves modulo those sheaves that have such a finite resolution. It is shown in [9] that this category is equivalent to the category of matrix factorisations; in particular, any object in D_{Sg} corresponds to a matrix factorisation, and the open string spectrum that is described in D_{Sg} in terms of morphisms of modules describing the branes, corresponds exactly to the BRST invariant spectrum of the Landau Ginzburg theory. More precisely, consider a Landau Ginzburg potential $W : \mathbb{C}^n \to \mathbb{C}$ with an isolated critical point at the origin. [In our case the relevant LG potentials are $W_{ADE} : \mathbb{C}^3 \to \mathbb{C}$, and thus describe precisely the singular hupersurfaces of (5.1).] Denoting the fiber of W over 0 by S_0, [9] allows us to establish a relation between $D_{Sg}(S_0)$ and the Landau-Ginzburg category. For this, we associate to any factorisation $W = d_0 d_1$

$$\left(\begin{tikzcd} P_1 \arrow[r, shift left, "d_1"] & P_0 \arrow[l, shift left, "d_0"] \end{tikzcd} \right)$$

the short exact sequence

$$0 \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow \text{Coker } d_1 \longrightarrow 0 \,. \tag{5.2}$$

The geometrical object associated to the factorisation is then the sheaf Coker d_1, which, since it is annihilated by W, is a sheaf on S_0. For further details on this functor and mathematical proofs, we refer to [9].

5.1 Singular geometry vs. Landau Ginzburg model

As we have just seen, the very same matrix factorisations that describe B-type D-branes in the Landau Ginzburg theory with superpotential W also characterise the B-type D-branes of the singular hypersurface $W = 0$ and its possible resolutions. Despite their formal similarities, these two description are however rather different: string theory on the singular geometry is, in sigma-model language, a singular limit of a theory with $c = 6$ that has no well-behaved conformal field theory description. On the other hand, the Landau Ginzburg model flows to a perfectly well defined conformal field theory, namely the $N = 2$ minimal model with $c = 3k/(k + 2)$.

From the closed string point of view the relation between singular geometries and $N = 2$ conformal field theories has been studied before in [21, 22, 23, 24]. In particular, characteristic properties of the Landau Ginzburg model, such as the central charge and the chiral ring, have been compared with concepts appearing in singularity theory (singularity index, local ring of W). The agreement of the matrix factorisations thus provides a natural extension of this correspondence to the open string sector. Since the matrix factorisations determine (the topological part of the) open string spectrum, this may even suggest that the B-type D-branes of the Landau-Ginzburg model (or of the $N = 2$ minimal model) provide a worldsheet description of the D-branes that become massless in the singular geometry.

5.2 Singular geometry vs. the orbifold description

String theory compactified on a singular surface does not have a well-defined perturbation theory since at the singular point non-perturbative (D-brane) states become massless.

However, it is possible to give a well-defined conformal field theory description for a closely related background where a non-zero B-field has been switched on for all singular cycles. (This B-field prevents the D-branes from becoming massless in the singular limit, and therefore avoids the breakdown of string perturbation theory [25, 26].) The relevant conformal field theory is the orbifold theory $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$. Orbifold theories are well behaved, and it is known how to describe their D-branes. In particular, following [10], the different D-branes are in essence characterised by the representation of the orbifold group Γ that acts on the corresponding Chan-Paton indices. The charges are then generated by the branes that are associated to the non-trivial irreducible representations of Γ. If we associate to each such representation a node of a (Dynkin) diagram, and connect nodes if the open string between the corresponding D-branes has a (massless) hypermultiplet in its spectrum, we recover precisely again the corresponding ADE Dynkin diagrams. Furthermore, the dimension of the irreducible representation of Γ equals the Kac-label of the corresponding node. This suggests, in particular, that the orbifold description captures at least some of the structure of the geometry described by $W = 0$.

Since the fundamental D-branes of the orbifold theory also give rise to the same ADE Dynkin diagram, they should be in natural one-to-one correspondence with the matrix factorisations of the LG potential.¶ In fact this relation can be understood fairly directly. As described in [27] following the ideas of [19, 20], for each irreducible representation of Γ one can find a module of the Γ-invariant part of $\mathbb{C}[X, Y]$. Unless the representation of Γ is trivial, these modules are not free, and the relations among the generators of the module can be described by a matrix. This is then precisely the matrix that appears as one factor of the matrix factorisation. For the two cases of interest in this paper, this can be done explicitly as follows.

5.2.1 A-type singularity

In the case of A-type singularities, the generator of the orbifold group acts on $\mathbb{C}^2$ as

$$gX = \xi X\,, \qquad gY = \xi^{-1} Y\,, \qquad \xi = e^{\frac{2\pi i}{n}}\,. \tag{5.3}$$

The singular surface can be described by the polynomial subring in the two variables X, Y that is invariant under the orbifold action. This invariant subring is generated by $x = XY$, $y = Y^n$ and $z = X^n$. These polynomials are however not independent, but satisfy the equation $x^n - yz = 0$, which is just the singular hypersurface equation of the A-type singularity.

The orbifold group Γ is in this case $\mathbb{Z}_n$, which has $n-1$ irreducible non-trivial representations. All of these representations are one-dimensional: for $l = 1, \ldots, n-1$ the corresponding representation associates to the generating element $g \in \Gamma$ the phase $e^{\frac{2\pi i l}{n}}$.

These $n-1$ representations are then in one-to-one correspondence to the matrix factorisations of the Landau-Ginzburg potential (2.13), or equivalently, the boundary states of the minimal model (2.15), whose label take values $L = l - 1 = 0, \ldots, k = n-2$.‖ As explained in [27] we can associate to each representation of Γ a module for the invariant

¶However, the open string spectra should not (and do not) agree: because of the resolution certain massless open string states of the Landau Ginzburg theory become massive in the orbifold theory.

‖The superpotential $W = x^n - yz$ (that is equivalent by a change of variables to $W = x^n - y^2 - z^2$)

part of the polynomial ring in two variables. For example, for the representation labelled by l the corresponding module is generated by X^l and Y^{n-l}. This is not a free module since we have the relations between the generators $s_1 = Y^{n-l}$, $s_2 = X^l$:

$$x^l s_1 - y s_2 = 0\,, \qquad -z s_1 + x^{n-l} s_2 = 0\,. \tag{5.4}$$

The matrix of relations

$$d_1 = \begin{pmatrix} x^l & -y \\ -z & x^{n-l} \end{pmatrix} \tag{5.5}$$

is then one of the matrices appearing in the matrix factorisation. Conversely, we can apply the functor of Orlov and recover the module (and thus the representation of Γ) from the factorisation.

5.2.2 D-type singularity

The D-type minimal model corresponds to the case where the orbifold is a binary dihedral group. Its two generators act on $\mathbb{C}^2$ as

$$g = \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{where } \beta = e^{\frac{\pi i}{n}} \tag{5.6}$$

and satisfy the relations

$$g^{2n} = \mathbf{1}\,, \qquad h^2 = g^n\,, \qquad hgh^{-1} = g^{-1}\,. \tag{5.7}$$

The ring of invariant polynomials under the group is generated by

$$x = (XY)^2\,, \qquad y = \frac{1}{2}(X^{2n} + Y^{2n})\,, \qquad z = \frac{i}{2}(XY)(X^{2n} - Y^{2n})\,, \tag{5.8}$$

which satisfy the D-type hypersurface equation $x^{n+1} - xy^2 - z^2 = 0$. Generically, the irreducible representations of this group are two-dimensional, with

$$\rho^{(l)}(g) = \begin{pmatrix} \beta^l & 0 \\ 0 & \beta^{-l} \end{pmatrix} \tag{5.9}$$

and

$$\rho^{(l)}(h) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ (l \text{ odd})\,, \qquad \rho^{(l)}(h) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \ (l \text{ even})\,. \tag{5.10}$$

Here $l = 0, 1, \ldots, 2n-1$, and the representations $\rho^{(l)}$ and $\rho^{(2n-l)}$ are equivalent. Furthermore, the representations $\rho^{(0)}$ and $\rho^{(n)}$ are not irreducible but can be further decomposed into one-dimensional representations: $\rho^{(0)}$ contains the trivial representation and the one where $g \to 1, h \to -1$. Likewise, $\rho^{(n)}$ can be decomposed into the two one-dimensional irreducible representations $g \to -1$ and $h \to \pm 1$ (l even) or $h \to \pm i$ (l odd).

differs from $W_A = x^n$ by changing the GSO-projection twice. One would therefore expect that these two superpotentials are equivalent. From the point of view of singularity theory this equivalence is known as Knörrer periodicity [28].

These representations are exactly in one-to-one correspondence with the boundary states of section 3.3, or equivalently the matrix factorisations of section 4.2. More explicitly, the representation $\rho^{(l)}$ corresponds precisely to the boundary state with $L = l$. Since $n = k/2$, the reducible representation $\rho^{(n)}$ corresponds then to the non-fundamental brane with $L = k/2$ which can be resolved into two boundary states (that correspond in turn to the two one-dimensional representations with $g \mapsto -1$). Also the identification $\rho^{(l)} \simeq \rho^{(2n-l)}$ mirrors the equivalence of boundary states (3.30). As in the A-case, one can also relate these representations to certain modules of the invariant algebra $\mathbb{C}[X,Y]^\Gamma$, and obtain the corresponding matrix factorisations in this manner (see [27]). It is maybe remarkable that the blow-ups that correspond to the one-dimensional representations of Γ are quite different to those that are associated to the two-dimensional representations.

For completeness we should also mention yet another class of models with $c = 6$ describing the motion of strings on ALE spaces. In this approach (which is due to [29]), the minimal models are tensored with another theory that contributes the missing central charge to get $c = 6$. To be more precise, [29] consider the tensor product of two coset theories $SL(2,\mathbb{R})/U(1) \times SU(2)/U(1)$, and orbifold by an appropriate discrete group to impose the charge integrality condition. In terms of Landau-Ginzburg potentials, these models can be written as

$$\begin{aligned} A_{n-1}: & \quad W = -\mu t^{-n} + x^n - yz = \; 0 \\ D_{n/2+1}: & \quad W = -\mu t^{-n} + x^{n/2} - xy^2 - z^2 = \; 0\,, \end{aligned} \tag{5.11}$$

which are again subject to an integer charge projection. As argued in [29], the value of the B-field in these models is 0, but now $\mu \neq 0$, and we are therefore discussing the resolved geometry. The D-branes of this model have been investigated in [30, 31]. If one wants to consider B-type D-branes in these models, one can effectively reduce the discussion to A-type D-branes in the minimal model part. The reason is that the orbifold procedure (followed by the mirror map) maps A-type to B-type branes, and, as noticed in [30, 31], the contribution of the non-compact coset can be captured by universal factors. Since this description involves effectively the A-type D-branes of the minimal model, the relation to matrix factorisations and branes in the singular geometry is however less evident.

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